Caldirola-Kanai Oscillator in Classical Formulation of Quantum Mechanics

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Abstract

The quadrature distribution for the quantum damped oscillator is introduced in the framework of the formulation of quantum mechanics based on the tomography scheme. The probability distribution for the coherent and Fock states of the damped oscillator is expressed explicitly in terms of Gaussian and Hermite polynomials, correspondingly.

In classical mechanics, the description of the motion with friction is described by the equation of motion has no ambiguities which are present in the quantum description. The quantum friction in the classical formulation of quantum mechanics was considered in [1]. The aim of this work is to discuss the problem of friction for the quantum Caldirola-Kanai oscillator [2, 3].

Moyal [4] obtained a evolution equation for quantum states in the form of the classical stochastic equation for the function which turned out to be the Wigner quasidistribution function [5] which can not be considered as probability since it takes negative values. Mancini, Man'ko and Tombesi [6] obtained the evolution equation for quantum state in the form of the classical stochastic equation for the function which turned out to be probability distribution for position measured in ensemble of squeezed and rotated reference frames in the classical phase space of system. The idea of this classical-like formulation of quantum dynamics uses the notion of optical tomography suggested by Vogel and Risken [7]. Man'ko obtained [8] the equation for energy levels in the framework of the classical-like formulation of quantum mechanics and rederived the energy spectrum of quantum oscillator (see also [9]).

The distribution $w\left(X,\mu,\nu,t\right)$ for the generic linear combination of quadratures, which is a measurable observable

$$\widehat{X}=\mu\widehat{q}+\nu\widehat{p}, \tag{1}$$

where $\widehat{q}$ and $\widehat{p}$ are the position and momentum, respectively, depending on two extra real parameters μ, ν is related to the state of the quantum system is expressed in terms of its Wigner function $W(q,p,t)$ as follows [6, 8]

$$w(X,\mu,\nu,t)=\int\exp\left[-ik\left(X-\mu q-\nu p\right)\right]W(q,p,t)\,\frac{dk\,dq\,dp}{(2\pi)^2}. \tag{2}$$

The distribution is normalized

$$\int w(X,\mu,\nu,t)\,d\,X=1. \tag{3}$$

As it was shown in [2, 3], the quantum friction appears system with the Hamiltonian (we assume $\hbar=m=1$)

$$\widehat{H}(t)=\frac{\widehat{p}^2}{2}\exp\left(-2\gamma t\right)+\omega^2\exp\left(2\gamma t\right)\frac{\widehat{q}^2}{2}, \tag{4}$$

where the friction coefficient γ and the frequency of the quantum oscillator ω are taken to be constant. For this system, the wave functions of the coherent $\mid\alpha\rangle$ and Fock $\mid n\rangle$ states can be written as [10] (we assume $\omega=1$)

$$\Psi_\alpha(q,t)=\frac{1}{\sqrt[4]{\pi}\sqrt{\varepsilon}}\exp\left(\frac{i\,\dot{\varepsilon}\,e^{2\gamma t}}{2\varepsilon}q^2+\frac{\sqrt{2}\alpha}{\varepsilon}q-\frac{\dot{\varepsilon}^*}{2\varepsilon}\alpha^2-\frac{\mid\alpha\mid^2}{2}\right), \tag{5}$$

$$\Psi_n(q,t)=\frac{1}{\sqrt[4]{\pi}\sqrt{\varepsilon}}\left(\frac{\varepsilon^*}{2\varepsilon}\right)^{\frac{n}{2}}\frac{1}{\sqrt{n!}}\exp\left(\frac{i\,\dot{\varepsilon}\,e^{2\gamma t}}{2\varepsilon}q^2\right)H_n\left(\frac{q}{\sqrt{\varepsilon\varepsilon^*}}\right). \tag{6}$$

In these formulas the time-dependent function $\varepsilon(t)$ satisfies the equation

$$\ddot{\varepsilon}\,(t)+2\gamma\,\dot{\varepsilon}\,(t)+\varepsilon(t)=0 \tag{7}$$

and the initial conditions

$$\varepsilon(0)=\frac{1}{\sqrt{\Omega}},\qquad\dot{\varepsilon}\,(0)=\frac{i\Omega-\gamma}{\sqrt{\Omega}}, \tag{8}$$

where $\Omega^2=1-\gamma^2$. The solution $\varepsilon(t)$ has the form

$$\varepsilon(t)=\frac{1}{\sqrt{\Omega}}e^{-\gamma t}\left[\cos\left(\Omega t\right)+i\sin\left(\Omega t\right)\right]. \tag{9}$$

The physical meaning of the Fock state of the Caldirola-Kanai oscillator (6) was discussed in [8]. It was shown that this state is a loss-energy state, and the wave function of this state has the property of periodicity in time with purely imaginary period. Using the known expression of Wigner function in terms of the wave function

of the coherent state (5) (see [6, 8]) and calculating the integral (2), we obtain the probability distribution for the coherent state

$$
\begin{aligned}
w_\alpha \;=\; & \frac{1}{\sqrt{\pi\varepsilon\varepsilon^*\left(a^2+b^2\right)}}\exp\left(-\mid\alpha\mid^2\right)\exp\left(-\frac{X^2}{\varepsilon\varepsilon^*\left(a^2+b^2\right)}\right)\\
& \otimes\exp\left[-\alpha^2\frac{\varepsilon^{*2}\left(a-ib\right)^2}{2\varepsilon\varepsilon^*\left(a^2+b^2\right)}+\alpha\frac{\sqrt{2}\varepsilon^*X\left(a-ib\right)}{\varepsilon\varepsilon^*\left(a^2+b^2\right)}\right]\\
& \otimes\exp\left[-\alpha^{*2}\frac{\varepsilon^2\left(a+ib\right)^2}{2\varepsilon\varepsilon^*\left(a^2+b^2\right)}+\alpha^*\frac{\sqrt{2}\varepsilon X\left(a+ib\right)}{\varepsilon\varepsilon^*\left(a^2+b^2\right)}\right]. \qquad (10)
\end{aligned}
$$

Analogously using the wave function (6), we find the probability distribution for the Fock state

$$
w_n\left(X,\mu,\nu,t\right)=w_0\left(X,\mu,\nu,t\right)\frac{1}{2^n n!}H_n^2\left(\frac{X}{\sqrt{\varepsilon\varepsilon^*\left(a^2+b^2\right)}}\right), \qquad (11)
$$

where the probability distribution of the oscillator ground-like state is

$$
w_0\left(X,\mu,\nu,t\right)=\frac{1}{\sqrt{\pi\varepsilon\varepsilon^*\left(a^2+b^2\right)}}\exp\left(-\frac{X^2}{\varepsilon\varepsilon^*\left(a^2+b^2\right)}\right) \qquad (12)
$$

and

$$
a=\frac{\exp\left(2\gamma t\right)\nu\left(\varepsilon^*\,\dot{\varepsilon}+\varepsilon\,\dot{\varepsilon^*}\right)}{2\varepsilon\varepsilon^*}+\mu, \qquad b=\frac{\nu}{\varepsilon\varepsilon^*}. \qquad (13)
$$

Here $\varepsilon\left(t\right)$ is given by equation (9). In Fig. 1 we show the probability distribution for the first excited state (loss-energy state) $w_1\left(X,\varphi,t\right)$ as a function of the rotation angle φ (abscissa) and the homodyne output variable X (ordinate) [7]

$$
\widehat{X}\left(\varphi\right)=\widehat{q}\cos\varphi-\widehat{p}\sin\varphi. \qquad (14)
$$

In Fig. 1 we assume $t=5$ and $\gamma=0.05$.

It was shown in [6] that for the system with Hamiltonian

$$
\widehat{H}(t)=\frac{\widehat{p}^2}{2}+\widehat{V}(q,t) \qquad (15)
$$

the quantum evolution equation alternative to the time-dependent Schrödinger equation has the form

$$
\dot{w}-\mu\frac{\partial}{\partial\nu}w-i\left[V\left(-\frac{1}{\partial/\partial X}\frac{\partial}{\partial\mu}-i\frac{\nu}{2}\frac{\partial}{\partial X},t\right)-V\left(-\frac{1}{\partial/\partial X}\frac{\partial}{\partial\mu}+i\frac{\nu}{2}\frac{\partial}{\partial X},t\right)\right]w=0. \qquad (16)
$$

For the damped oscillator this equation takes the form [1]

$$
\dot{w}-\mu\frac{\partial}{\partial\nu}w-i\left[\widetilde{V}\left(-\frac{1}{\partial/\partial X}\frac{\partial}{\partial\mu}-i\frac{\nu}{2}\frac{\partial}{\partial X},t'\right)-\widetilde{V}\left(-\frac{1}{\partial/\partial X}\frac{\partial}{\partial\mu}+i\frac{\nu}{2}\frac{\partial}{\partial X},t'\right)\right]w=0, \qquad (17)
$$

where

$$\widetilde{V}\left(q,t'\right)=\exp\left[2\gamma t\left(t'\right)\right]V\left[q,t\left(t'\right)\right]=\exp\left[4\gamma t\left(t'\right)\right]\frac{q^{2}}{2},\tag{18}$$

$$t'\left(t\right)=\frac{1-\exp(-2\gamma t)}{2\gamma},\qquad t\left(t'\right)=-\frac{\ln(1-2\gamma t')}{2\gamma}\tag{19}$$

and

$$\frac{\partial t\left(t'\right)}{\partial t'}=\exp\left(2\gamma t\right).\tag{20}$$

The dot means partial derivative in t'. Using the relation (18), one can rewrite (17) as

$$\frac{\partial}{\partial t'}w-\mu\frac{\partial}{\partial\nu}w+\exp\left(4\gamma t\right)\nu\frac{\partial}{\partial\mu}w=0.\tag{21}$$

One can check that the probability distributions w_α (10) and w_n (11) satisfy this equation.

Let us consider invariants of the damped quantum oscillator $\widehat{a}^{\dagger}\widehat{a}\left(t\right)$, $\left(\widehat{a}^{\dagger}\widehat{a}\right)^{*}\left(t\right)$ in the classical formulation of quantum mechanics. Here asterisk means the complex conjugate operator. The operator $\widehat{a}^{\dagger}\widehat{a}\left(t\right)$ acts on the variable q, and the operator $\left(\widehat{a}^{\dagger}\widehat{a}\right)^{*}\left(t\right)$ acts on the variable q' of the density matrix $\rho_n\left(q,q',t\right)$, which describes the Fock state $\mid n\rangle$ of the system. These invariants act on the distribution w_n of the Fock state (11) as

$$\widehat{a}^{\dagger}\widehat{a}\left(t\right)w_n\left(X,\mu,\nu,t\right)=nw_n\left(X,\mu,\nu,t\right),\tag{22}$$

$$\left(\widehat{a}^{\dagger}\widehat{a}\right)^{*}\left(t\right)w_n\left(X,\mu,\nu,t\right)=nw_n\left(X,\mu,\nu,t\right).\tag{23}$$

The invariants $\widehat{a}^{\dagger}\widehat{a}\left(t\right)$ and $\left(\widehat{a}^{\dagger}\widehat{a}\right)^{*}\left(t\right)$ have the following form

$$\begin{aligned}a^{\dagger}a\left(t\right)\quad=\quad&\frac{1}{2}\Bigg\{\left(\frac{\partial}{\partial X}\right)^{-2}\left[\varepsilon\varepsilon^{*}\left(\frac{\partial}{\partial\nu}\right)^{2}+\dot{\varepsilon}\dot{\varepsilon}^{*}\,e^{4\gamma t}\left(\frac{\partial}{\partial\mu}\right)^{2}-e^{2\gamma t}\left(\varepsilon^{*}\,\dot{\varepsilon}+\varepsilon\,\dot{\varepsilon}^{*}\right)\frac{\partial^{2}}{\partial\mu\partial\nu}\right]\\&-\left(\frac{\partial}{\partial X}\right)^{2}\left[\varepsilon\varepsilon^{*}\mu^{2}+\dot{\varepsilon}\dot{\varepsilon}^{*}\,e^{4\gamma t}\nu^{2}+e^{2\gamma t}\left(\varepsilon^{*}\,\dot{\varepsilon}+\varepsilon\,\dot{\varepsilon}^{*}\right)\mu\nu\right]\\&+i\left[\frac{\varepsilon\varepsilon^{*}}{2}\left(\mu\frac{\partial}{\partial\nu}+\frac{\partial}{\partial\nu}\mu\right)+\frac{\dot{\varepsilon}^{*}\,\varepsilon e^{2\gamma t}}{2}\nu\frac{\partial}{\partial\nu}+\frac{\varepsilon^{*}\,\dot{\varepsilon}\,e^{2\gamma t}}{2}\frac{\partial}{\partial\nu}\nu\right]\\&-i\left[\frac{\dot{\varepsilon}\dot{\varepsilon}^{*}}{2}\left(\nu\frac{\partial}{\partial\mu}+\frac{\partial}{\partial\mu}\nu\right)+\frac{\varepsilon^{*}\,\dot{\varepsilon}\,e^{2\gamma t}}{2}\mu\frac{\partial}{\partial\mu}+\frac{\dot{\varepsilon}^{*}\,\varepsilon e^{2\gamma t}}{2}\frac{\partial}{\partial\mu}\mu\right]\Bigg\}\end{aligned}\tag{24}$$

and

$$\begin{aligned}\left(a^{\dagger}a\right)^{*}\left(t\right)\quad=\quad&\frac{1}{2}\Bigg\{\left(\frac{\partial}{\partial X}\right)^{-2}\left[\varepsilon\varepsilon^{*}\left(\frac{\partial}{\partial\nu}\right)^{2}+\dot{\varepsilon}\dot{\varepsilon}^{*}\,e^{4\gamma t}\left(\frac{\partial}{\partial\mu}\right)^{2}+e^{2\gamma t}\left(\varepsilon^{*}\,\dot{\varepsilon}+\varepsilon\,\dot{\varepsilon}^{*}\right)\frac{\partial^{2}}{\partial\mu\partial\nu}\right]\\&-\left(\frac{\partial}{\partial X}\right)^{2}\left[\varepsilon\varepsilon^{*}\mu^{2}+\dot{\varepsilon}\dot{\varepsilon}^{*}\,e^{4\gamma t}\nu^{2}-e^{2\gamma t}\left(\varepsilon^{*}\,\dot{\varepsilon}+\varepsilon\,\dot{\varepsilon}^{*}\right)\mu\nu\right]\end{aligned}$$

$$
-i\left[\frac{\varepsilon\varepsilon^*}{2}\left(\mu\frac{\partial}{\partial\nu}+\frac{\partial}{\partial\nu}\mu\right)-\frac{\varepsilon^*\,\dot\varepsilon\,e^{2\gamma t}}{2}\nu\frac{\partial}{\partial\nu}-\frac{\dot\varepsilon^*\,\varepsilon e^{2\gamma t}}{2}\frac{\partial}{\partial\nu}\nu\right]
$$

$$
+i\left[\frac{\dot\varepsilon\dot\varepsilon^*}{2}\left(\nu\frac{\partial}{\partial\mu}+\frac{\partial}{\partial\mu}\nu\right)-\frac{\dot\varepsilon^*\,\varepsilon e^{2\gamma t}}{2}\mu\frac{\partial}{\partial\mu}-\frac{\varepsilon^*\,\dot\varepsilon\,e^{2\gamma t}}{2}\frac{\partial}{\partial\mu}\mu\right]\Bigg\}. \qquad (25)
$$

To obtain this form of the operators under discussion, we used the correspondence of the action of the operators on the Wigner function $W(q,p,t)$ and the probability distribution $w(X,\mu,\nu,t)$ [8].

The main result of this work is the introduction of a positive normalized distribution function (probability distribution) for the description of the quantum states of the damped quantum oscillator. This distribution contains complete information about the state of system. For the probability distribution of the damped oscillator the quantum evolution equation is found, which is an alternative to the Schrödinger equation.

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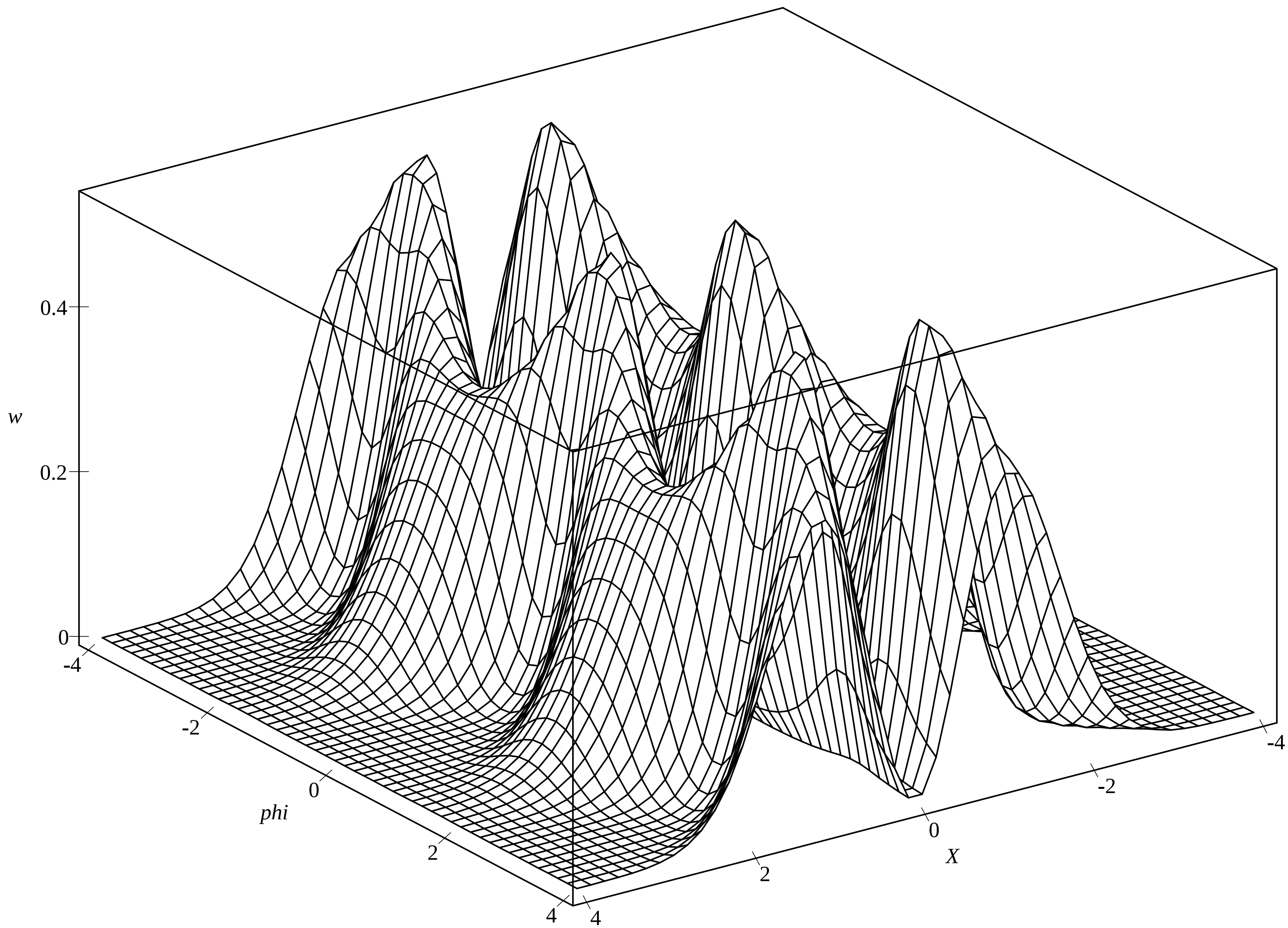
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